Distribution Functions

The calculation of the pressure of a gas in the CCR unit *Kinetic Theory* gives us interesting information about the average square speed, and therefore the average energies, of the molecules in a gas, but it does not yield any details about the *distribution* of molecular velocities. Distribution functions are encountered frequently in Chapter 8 and in several later chapters. Here we will discuss distribution functions in general, with some elementary examples from common experience.

Suppose a teacher gave a 25-point quiz to a large number N of students. In order to describe the results of the quiz, the teacher might give the average or median score, but this would not be a complete description. For example, if all N students received 12.5, this is quite a different result than if $N/2$ students received 25 and $N/2$ received 0, though both results have the same average. A more complete description would be to give the number n_i who received the score s_i for all scores s_i between 0 and 25. An alternative would be to divide n_i by the total number of students N to give the fraction of students $f_i = n_i/N$ receiving the score s_i. Both n_i and f_i (which depend on the variable s) are called *distribution functions*. The fractional distribution f_i is slightly more convenient to use. The probability that one of the N students selected at random received the score s_i equals the number of students that received that score, $n_i = Nf_i$, divided by the total number N; thus this probability equals the distribution function f_i. Note that

$$\sum_i f_i = \sum_i \frac{n_i}{N} = \frac{1}{N}\sum_i n_i$$

and since

$$\sum_i n_i = N$$

we have

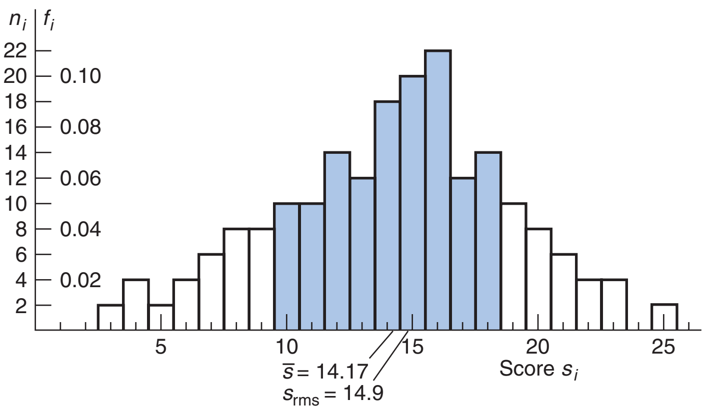
$$\sum_i f_i = 1 \qquad \textbf{DF-1}$$

Equation DF-1 is called the *normalization condition* for fractional distribution functions. A possible distribution function for a 25-point quiz is shown in Figure DF-1.

To find the average score, all the scores are added and the result is divided by N. Since each score s_i was obtained by $n_i = Nf_i$ students, this procedure is equivalent to

$$\bar{s} = \frac{1}{N}\sum_i s_i n_i = \sum_i s_i f_i \qquad \textbf{DF-2}$$

We will take Equation DF-2 as the definition of the *average* (or *mean*) score $\bar{s}$. Similarly, the average of any function $g(s)$ is defined by



DF-1 Grade distribution for a 25-point quiz given to 200 students; n_i is the number and $f_i = n_i/N$ is the fraction of students receiving the score s_i. The average score $\bar{s}$ and the root-mean-square score s_{rms} are indicated. The shaded area indicates the scores within 1 standard deviation of the mean.

$$\overline{g(s)} = \sum_i g(s_i) f_i \tag{DF-3}$$

In particular, the mean square score is often useful:

$$\overline{s^2} = \sum_i s_i^2 f_i$$

A useful quantity characterizing a distribution is the *standard deviation*, σ, defined by

$$\sigma = \left[\sum_i (s_i - \bar{s})^2 f_i \right]^{1/2} \tag{DF-4}$$

Note that

$$\sum_i (s_i - \bar{s})^2 f_i = \sum_i s_i^2 f_i + \bar{s}^2 \sum_i f_i - 2\bar{s} \sum_i s_i f_i = \overline{s^2} - \bar{s}^2$$

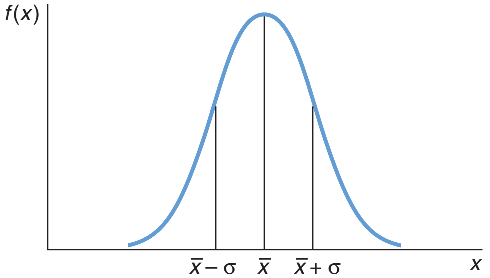
Therefore

$$\sigma = (\overline{s^2} - \bar{s}^2)^{1/2} \tag{DF-5}$$

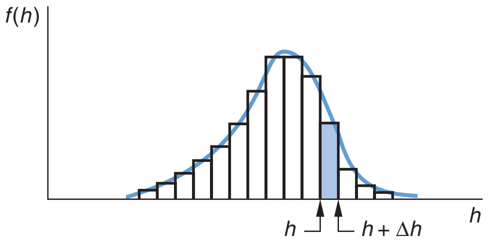
The standard deviation measures the spread of the values s_i about the mean. For most distributions there will be few values that differ from $\bar{s}$ by more than a few multiples of σ. *In the case of the normal Gaussian distribution, common in the theory of errors, about two-thirds of the values lie within $\pm\sigma$ of the mean value.* A Gaussian distribution is shown in Figure DF-2.

If a student were selected at random from the class and one had to guess that student's score, the best guess would be the score obtained by the greatest number of students, called the most probable score, s_m. For the distribution in Figure DF-1, s_m is 16 and the average score, $\bar{s}$, is 14.17. The root-mean-square score, $s_{\text{rms}} = (\overline{s^2})^{1/2}$, is 14.9, and the standard deviation σ is 4.6. Note that 66 percent of the scores for this distribution lie within $\bar{s} \pm \sigma = 14.17 \pm 4.6$.

Now consider the case of a continuous distribution. Suppose we wanted to know the distribution of heights of a large number of people. For a finite number N, the



DF-2 Gaussian or normal distribution curve. The curve is symmetrical about the mean value $\bar{x}$, which is also the most probable value. Sixty-eight percent of the area under the curve is within 1 standard deviation of the mean. This curve describes the distribution of random errors in many experimental situations.



DF-3 A possible height distribution. The fraction of the number of heights between h and $h + \Delta h$ is proportional to the shaded area. The histogram can be approximated by a continuous curve as shown.

number of persons *exactly* 6 feet tall would be zero. If we assume that height can be determined to any desired accuracy, there is an infinite number of possible heights, and the chance that anybody has a particular exact height is zero. We would therefore divide the heights into intervals Δh (for example, Δh could be 0.1 ft) and ask what fraction of people have heights that fall in any particular interval. This number depends on the size of the interval. We define the distribution function $f(h)$ as the fraction of the number of people with heights in a particular interval divided by the size of the interval. Thus, for N people, $Nf(h)\Delta h$ is the number of people whose height is in the interval between h and $h + \Delta h$. A possible height-distribution function is plotted in Figure DF-3. The fraction of people with heights in a particular interval is the area of the rectangle $\Delta h \times f(h)$. The total area represents the sum of all fractions; thus it must equal 1. If N is very large, we can choose Δh very small and still have $f(h)$ vary only slightly between intervals. The histogram $f(h)$ versus h approaches a smooth curve as $N \to \infty$ and $\Delta h \to 0$. In many cases of importance, the number of objects N is extremely large and the intervals can be taken as small as measurement allows. The distribution functions $f(h)$ are usually considered to be continuous functions, intervals are written dh, and the sums are replaced by integrals. For example, if $f(h)$ is a continuous function, the average height, which we will write as $\langle h \rangle$ for the continuous function $f(h)$, is[1]

$$\langle h \rangle = \int h f(h)\, dh \qquad \textbf{DF-6}$$

and the normalization condition expressing the fact that the sum of all fractions is 1 is

$$\int f(h)\, dh = 1 \qquad \textbf{DF-7}$$

The CCR unit *Boltzmann Distribution* illustrates a physical application of distribution functions as does the computation of expectation values in Chapters 6 and 7, among others, and in a number of More sections.

[1]The limits on the integration depend on the range of the variable. For this case, h ranges from 0 to ∞. We shall often omit explicit indication of the limits when the range of the variable is clear.